

INDUCED *-REPRESENTATIONS AND C^* -ENVELOPES OF SOME QUANTUM *-ALGEBRAS.

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ABSTRACT. We consider three quantum algebras: the q -oscillator algebra, the Podles' sphere and the q -deformed enveloping algebra of $su(2)$. To each of these *-algebras we associate certain partial dynamical system and perform the "Mackey analysis" of *-representations developed in [SS]. As a result we get the description of "standard" irreducible *-representations. Further, for each of these examples we show the existence of a " C^* -envelope" which is canonically isomorphic to the covariance C^* -algebra of the partial dynamical system. Finally, for the q -oscillator algebra and the q -deformed $\mathcal{U}(su(2))$ we show the existence of "bad" representations.

INTRODUCTION AND PRELIMINARIES

The aim of this paper is to demonstrate a unified approach to the *-representation theory of various quantum algebras based on the techniques developed in [SS]. Most of the quantum *-algebras (e.g. non-compact quantum groups) possess unbounded *-representations. The main problem in the theory of unbounded *-representations is to define and classify the "well-behaved" *-representations of a given *-algebra. We recall two classical examples.

Example. Let \mathfrak{g} be a finite-dimensional real Lie algebra, G be the corresponding simply connected Lie group and $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ be the complex enveloping *-algebra of \mathfrak{g} . A *-representation π of \mathfrak{g} is called *integrable* if $\pi = dU$ for some unitary representation U of G . If $G \neq \mathbb{R}$ there exists a *-representation of \mathfrak{g} which is not integrable and, moreover, cannot be extended to an integrable representation even in a larger Hilbert space, see [S1]. Already in the case $G = \mathbb{R}^2$, $\mathfrak{g} = \mathbb{C}[x_1, x_2]$ the category of all *-representations of \mathfrak{g} is in a certain sense "very large" as shown in [S3, Section 9].

Example. Let W_n be the n -dimensional Weyl algebra. That is, W_n is a complex *-algebra generated by self-adjoint elements p_i, q_i , $i = 1, \dots, n$, satisfying $[p_i, q_j] = -\delta_{ij}\mathbf{i}$, $[p_i, p_j] = [q_i, q_j] = 0$. A *-representation π of W_n is called *integrable* if $P_i = \pi(p_i)$, $Q_j = \pi(q_j)$, $i, j = 1, \dots, n$ are self-adjoint and the one-parameter unitary groups e^{itP_i} , $e^{is_jQ_j}$ satisfy the Weyl commutation relations. Already for W_1 one

Date: March 8, 2013.

2000 Mathematics Subject Classification. Primary 20G42, 47L60, 17B37; Secondary 16G99, 22D30, 16W50, 47L65.

Key words and phrases. Induced representations, group graded algebras, well-behaved representations, partial action of a group, Mackey analysis, C^* -envelope, q -deformed enveloping algebra, Podles' sphere, q -oscillator.

The first author was supported by the International Max Planck Research School for Mathematics in the Sciences (Leipzig).

can show the existence of “bad representations” and show that the category of all $*$ -representations is again “very large”, whereas the only integrable $*$ -representations are sums of copies of the Schrödinger representation.

We investigate the following three $*$ -algebras in details: the q -oscillator algebra \mathcal{A}_q for $q > 0$, the q -deformed enveloping algebra $\mathcal{U}_q(su(2))$, $q > 0$ and the Podles’ spheres $\mathcal{O}(S_{qr}^2)$, $q \in (0, 1)$, $r \in (0, \infty)$. The algebras \mathcal{A}_q and $\mathcal{U}_q(su(2))$ are deformations of W_1 and $\mathcal{U}_{\mathbb{C}}(su(2))$ respectively, however, for both these algebras the notion of “integrability” cannot be generalized in a direct way. Instead of this we use the approach from [SS], which applies to all three algebras \mathcal{A}_q , $\mathcal{O}(S_{qr}^2)$, $\mathcal{U}(su(2))$ as well as to their classical analogues. Let \mathcal{A} denote one of these algebras. The basic idea is to find a natural \mathbb{Z} -grading \mathcal{A}_k , $k \in \mathbb{Z}$, for \mathcal{A} such that $\mathcal{A}_0 =: \mathcal{B}$ is commutative. Further, we define the “positive” spectrum $\widehat{\mathcal{B}}^+$ of \mathcal{B} as the set of those characters $\chi \in \widehat{\mathcal{B}}$ which satisfy $\chi(a^*a) \geq 0$ for all $a \in \mathcal{A}$, such that $a^*a \in \mathcal{B}$. The group grading of \mathcal{A} defines a structure of a $*$ -algebraic bundle in the sense of [FD], and there is a canonical partial action α of \mathbb{Z} on $\widehat{\mathcal{B}}^+$. By means of the partial dynamical system $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$ we

- define well-behaved $*$ -representations,
- show that the irreducible ones naturally correspond to the orbits of $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$,
- construct the dual partial action β on $C_0(\widehat{\mathcal{B}}^+)$ and the partial crossed product C^* -algebra $C_0(\widehat{\mathcal{B}}^+) \times_{\beta} \mathbb{Z}$ in the sense of [Ex]; using the Woronowicz’s theory of affiliated operators, we establish a Morita equivalence between $C_0(\widehat{\mathcal{B}}^+) \times_{\beta} \mathbb{Z}$ and \mathcal{A} .

It turns out that every irreducible well-behaved representation of \mathcal{A} is induced from a one-dimensional representation. Thereby, the induction procedure is the generalized Rieffel induction introduced and studied in [SS]. This result can be viewed as an analogue of the following theorem by Kirillov (see [Kir]): Every irreducible unitary representation of a nilpotent Lie group is induced from a one-dimensional representation of a certain subgroup.

In each of three cases the constructed crossed product C^* -algebra is of a special kind. Namely, the partial action of \mathbb{Z} on $C_0(\widehat{\mathcal{B}}^+)$ is generated by a single partial automorphism Θ , see [Ex, McC] and Section 0.2. In this case the partial crossed product C^* -algebra coincides with the covariance C^* -algebra of the partial automorphism in the sense of [Ex, Definition 3.7]. In the case $\mathcal{A} = \mathcal{O}(S_{qr}^2)$ all $*$ -representations are bounded, hence well-behaved, and the crossed product C^* -algebra $C_0(\widehat{\mathcal{B}}^+) \times_{\alpha} \mathbb{Z}$ is isomorphic to the enveloping C^* -algebra of $C_{env}^*(\mathcal{A})$.

Finally, for \mathcal{A}_q and for $\mathcal{U}_q(su(2))$ we show the existence of “bad” representations. More precisely, we prove the existence of a $*$ -representation which is not well-behaved and cannot be extended to a well-behaved $*$ -representation even in a larger Hilbert space. It generalizes the well-known results for W_1 and $\mathcal{U}(su(2))$.

Among the examples which can be analyzed in the same spirit include various bounded and unbounded $*$ -algebras: quantum group algebras $SU_q(2)$, $SU_q(1, 1)$, q -deformed $\mathcal{U}(su(1, 1))$, different deformations of CAR and CCR, AF pre- C^* -algebras (see [Ex1]) etc.

0.1. $*$ -Algebras and $*$ -representations. By a $*$ -algebra we mean a complex associative algebra \mathcal{A} equipped with a mapping $a \mapsto a^*$ of \mathcal{A} into itself, called the *involution* of \mathcal{A} , such that $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$, $(ab)^* = b^* a^*$ and $(a^*)^* = a$ for

$a, b \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$. The unit of \mathcal{A} (if it exists) will be denoted by $1_{\mathcal{A}}$ or simply by 1 . For every *-algebra \mathcal{A} denote by $\sum \mathcal{A}^2$ the set of finite sums $\sum a_i^* a_i$, $a_i \in \mathcal{A}$.

Throughout this paper we use some terminology and results from unbounded representation theory in Hilbert space (see e.g. [S3]). We repeat some basic notions and facts. If T is a Hilbert space operator, $\mathcal{D}(T)$, \overline{T} and T^* denote its domain, its closure and its adjoint, respectively. Let \mathcal{D} be a dense linear subspace of a Hilbert space \mathcal{H} with scalar product $\langle \cdot, \cdot \rangle$. A *-representation of a *-algebra \mathcal{A} on \mathcal{D} is an algebra homomorphism π of \mathcal{A} into the algebra $L(\mathcal{D})$ of linear operators on \mathcal{D} such that $\langle \pi(a)\varphi, \psi \rangle = \langle \varphi, \pi(a^*)\psi \rangle$ for all $\varphi, \psi \in \mathcal{D}$ and $a \in \mathcal{A}$. We call $\mathcal{D}(\pi) := \mathcal{D}$ the domain of π and write $\mathcal{H}_{\pi} := \mathcal{H}$. Two *-representations π_1 and π_2 of \mathcal{A} are (unitarily) equivalent if there exists an isometric linear mapping U of $\mathcal{D}(\pi_1)$ onto $\mathcal{D}(\pi_2)$ such that $\pi_2(a) = U\pi_1(a)U^{-1}$ for $a \in \mathcal{A}$. The direct sum representation $\pi_1 \oplus \pi_2$ acts on the domain $\mathcal{D}(\pi_1) \oplus \mathcal{D}(\pi_2)$ by $(\pi_1 \oplus \pi_2)(a) = \pi_1(a) \oplus \pi_2(a)$, $a \in \mathcal{A}$. A *-representation π is called irreducible if a direct sum decomposition $\pi = \pi_1 \oplus \pi_2$ is only possible when $\mathcal{D}(\pi_1) = \{0\}$ or $\mathcal{D}(\pi_2) = \{0\}$. For a *-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ we denote by $\text{Res}_{\mathcal{B}}\pi$ its restriction to \mathcal{B} . The graph topology of π is the locally convex topology on the vector space $\mathcal{D}(\pi)$ defined by the norms $\varphi \mapsto \|\varphi\| + \|\pi(a)\varphi\|$, where $a \in \mathcal{A}$. If $\overline{\mathcal{D}(\pi)}$ denotes the completion of $\mathcal{D}(\pi)$ in the graph topology of π , then $\overline{\pi}(a) := \overline{\pi(a)} \upharpoonright \overline{\mathcal{D}(\pi)}$, $a \in \mathcal{A}$, defines a *-representation of \mathcal{A} with domain $\overline{\mathcal{D}(\pi)}$, called the closure of π . In particular, π is closed if and only if $\mathcal{D}(\pi)$ is complete in the graph topology of π . A *-representation π is called non-degenerate if $\pi(\mathcal{A})\mathcal{D}(\pi) := \text{Lin} \{ \pi(a)\varphi; a \in \mathcal{A}, \varphi \in \mathcal{D}(\pi) \}$ is dense in $\mathcal{D}(\pi)$ in the graph topology of π . If \mathcal{A} is unital and π is non-degenerate, then we have $\pi(1_{\mathcal{A}})\varphi = \varphi$ for all $\varphi \in \mathcal{D}(\pi)$. We say that π is cyclic if there exists a vector $\varphi \in \mathcal{D}(\pi)$ such that $\pi(\mathcal{A})\varphi$ is dense in $\mathcal{D}(\pi)$ in the graph topology of π . For a C*-algebra \mathfrak{A} and Hilbert space \mathcal{H} , denote by $\text{Rep}(\mathfrak{A}, \mathcal{H})$ the category of non-degenerate *-representations of \mathfrak{A} on \mathcal{H} . By $\text{Rep}\mathfrak{A}$ denote the category of all non-degenerate *-representations of \mathfrak{A} .

We recall the induction procedure for *-representations of general *-algebras developed in [SS, Section 2] in a slightly more general context. However, we will not perform this procedure but use Proposition 1.2 to get the explicit formulas. Let $\mathcal{B} \subseteq \mathcal{A}$ be *-algebras. A linear map $p : \mathcal{A} \rightarrow \mathcal{B}$ is called a bimodule projection if $p(a^*) = p(a)^*$, $p(b_1 a b_2) = b_1 p(a) b_2$, $p(1_{\mathcal{A}}) = 1_{\mathcal{B}}$, for all $a \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$. Let ρ be a *-representation of \mathcal{B} . Denote by $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{D}(\rho)$ the quotient of $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{B}$ by the linear span of vectors $ab \otimes \varphi - a \otimes \rho(b)\varphi$, $a \in \mathcal{A}$, $b \in \mathcal{B}$, $\varphi \in \mathcal{D}(\rho)$. We say that ρ is inducible from \mathcal{B} to \mathcal{A} via p if the sesquilinear form

$$(0.1) \quad \left\langle \sum_k x_k \otimes \varphi_k, \sum_l y_l \otimes \psi_l \right\rangle_0 := \sum_{k,l} \langle \rho(p(y_l^* x_k)) \varphi_k, \psi_l \rangle,$$

is positive semi-definite on $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{D}(\rho)$. Denote by \mathcal{K}_{ρ} the kernel of $\langle \cdot, \cdot \rangle_0$. Then $\mathcal{D}_0 = \mathcal{A} \otimes_{\mathcal{B}} \mathcal{D}(\rho) / \mathcal{K}_{\rho}$ is an inner-product space. Define a *-representation π on \mathcal{D}_0 via

$$\pi(a) \left(\sum_i [a_i \otimes \varphi_i] \right) := \sum_i [a a_i \otimes \varphi_i],$$

where $\sum_i [a_i \otimes \varphi_i] \in \mathcal{D}_0$ denotes the image of $\sum_i a_i \otimes \varphi_i$ under the quotient mapping. Finally define $\text{Ind}\rho$ to be the closure of π .

Our major application of the induction procedure will be in the following context. Let G be a discrete group and \mathcal{A} be a G -graded *-algebra. That is, \mathcal{A} is a direct

sum of vector spaces \mathcal{A}_g , $g \in G$, such that

$$(0.2) \quad \mathcal{A}_g \cdot \mathcal{A}_h \subseteq \mathcal{A}_{g \cdot h} \text{ and } (\mathcal{A}_g)^* \subseteq \mathcal{A}_{g^{-1}} \text{ for } g, h \in G.$$

The elements of $\bigcup_{g \in G} \mathcal{A}_g$ are called *homogeneous*. For every subgroup $H \subseteq G$ the sum $\bigoplus_{g \in H} \mathcal{A}_g =: \mathcal{A}_H$ is a $*$ -subalgebra of \mathcal{A} and the canonical projection $p : \mathcal{A} \rightarrow \mathcal{A}_H$ is a bimodule projection. If \mathcal{A}_e is commutative then a character $\chi : \mathcal{A}_e \rightarrow \mathbb{C}$ is inducible (via $p_e : \mathcal{A} \rightarrow \mathcal{A}_e$) if and only if $\chi(a^*a) \geq 0$ for all homogeneous $a \in \mathcal{A}$.

0.2. Partial actions and partial crossed products. The constructions and results of this subsection are taken from [Ex, McC]. A *partial action* of a discrete group G on a set X is a pair

$$\alpha = (\{\mathcal{D}_g\}_{g \in G}, \{\alpha_g\}_{g \in G}),$$

where $\mathcal{D}_g \subseteq X$, $g \in G$ are subsets and $\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g$ are bijections such that

- (i) $\alpha_g(\mathcal{D}_{g^{-1}} \cap \mathcal{D}_h) = \mathcal{D}_{gh} \cap \mathcal{D}_g$, $g, h \in G$,
- (ii) $\alpha_{hg}(x) = \alpha_h(\alpha_g(x))$, $x \in \mathcal{D}_{g^{-1}} \cap \mathcal{D}_{g^{-1}h^{-1}}$,
- (iii) $\mathcal{D}_e = X$, $\alpha_e = \text{Id}_X$.

For a partial action $\alpha = (\{\mathcal{D}_g\}_{g \in G}, \{\alpha_g\}_{g \in G})$ on a topological space X we require in addition that \mathcal{D}_g are open sets and $\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g$, $g \in G$ are homeomorphisms. We call (X, G, α) a *partial dynamical system* (p.d.s.).

For a partial action $\beta = (\{I_g\}_{g \in G}, \{\beta_g\}_{g \in G})$ of G on a C^* -algebra \mathfrak{B} we require in addition that I_g , $g \in G$ are closed two-sided ideals and $\beta_g : I_{g^{-1}} \rightarrow I_g$ are $*$ -isomorphisms. We call (\mathfrak{B}, G, β) a *partial C^* -dynamical system* (C^* -p.d.s.). For a p.d.s. (X, G, α) where X is a locally compact Hausdorff space we define the *dual C^* -p.d.s.* as follows. Put $\mathfrak{B} = C_0(X)$, $I_g = C_0(\mathcal{D}_g)$ and define $\beta_g : I_{g^{-1}} \rightarrow I_g$ by

$$(\beta_g(f))(x) = f(\alpha_{g^{-1}}(x)), \quad x \in \mathcal{D}_g, f \in I_{g^{-1}}, g \in G.$$

Direct computations show that $\beta = (\{I_g\}_{g \in G}, \{\beta_g\}_{g \in G})$ is a partial action on \mathfrak{B} and that (\mathfrak{B}, G, β) is a C^* -p.d.s.

Let (\mathfrak{B}, G, β) , $\beta = (\{I_g\}_{g \in G}, \{\beta_g\}_{g \in G})$ be a C^* -p.d.s. The *partial crossed product C^* -algebra* $\mathfrak{A} = \mathfrak{B} \times_{\beta} G$ is the enveloping C^* -algebra of the $*$ -algebra $\mathfrak{B}G$ defined as follows. $\mathfrak{B}G \subseteq \mathfrak{B} \otimes \mathbb{C}[G]$ is the linear span of the set $\{a \otimes g \mid a \in I_g\}$, with multiplication and involution defined by

$$(a \otimes g)(b \otimes h) := \alpha_g(\alpha_{g^{-1}}(a)b) \otimes gh, \quad (a \otimes g)^* := \alpha_{g^{-1}}(a^*) \otimes g^{-1}.$$

The examples of C^* -p.d.s. which appear below are of a special kind. Recall [Ex], that a *partial automorphism* of a C^* -algebra \mathfrak{B} is a triple $\Theta = (\theta, I, J)$, where $I, J \subseteq \mathfrak{B}$ are closed two-sided ideals and $\theta : I \rightarrow J$ is a $*$ -isomorphism. Set $I_0 = \mathfrak{B}$ and define I_n , $n \in \mathbb{Z}$, by induction

$$\begin{aligned} I_{n+1} &= \{a \in J \mid \theta^{-1}(a) \in I_n\}, \text{ for } n \geq 0, \\ I_{n-1} &= \{a \in I \mid \theta(a) \in I_n\}, \text{ for } n \leq 0. \end{aligned}$$

In particular, $I = I_{-1}$ and $J = I_1$. It can be checked, see [Ex, Section 3], that the triple $(\mathfrak{B}, \mathbb{Z}, \beta)$, where $\beta = (\{I_n\}_{n \in \mathbb{Z}}, \{\theta^n\}_{n \in \mathbb{Z}})$ is a C^* -p.d.s. The partial crossed product algebra $\mathfrak{B} \times_{\beta} \mathbb{Z}$ is called the *covariance algebra* of (\mathfrak{B}, Θ) and is denoted by $C^*(\mathfrak{B}, \Theta)$. As in the case of a crossed-product by a $*$ -automorphism, $*$ -representations of $C^*(\mathfrak{B}, \Theta)$ are in one-to-one correspondence with covariant representations of the pair (\mathfrak{B}, Θ) , see [Ex, Section 5]. In case of the C^* -p.d.s. defined by (\mathfrak{B}, Θ) a *covariant representation* $\pi \times u$ consists of a $*$ -representation

$\pi : \mathfrak{B} \rightarrow B(\mathcal{H})$ and a partial isometry u , whose initial and final spaces are $\overline{\pi(I)\mathcal{H}}$ and $\pi(J)\mathcal{H}$ respectively, so that

$$\pi(\theta(b)) = u\pi(b)u^*, \text{ holds for every } b \in I.$$

If the latter is satisfied, then $\pi \times u$ becomes a *-representation of $\mathfrak{B}\mathbb{Z}$, hence of $C^*(\mathfrak{B}, \mathbb{Z})$, via

$$(\pi \times u)(f \otimes k) = \pi(f)u^k, \text{ for } f \otimes k \in \mathfrak{B}\mathbb{Z},$$

where $u^{-k} = u^{*k}$ for $k \in \mathbb{N}$.

0.3. Unbounded elements affiliated with C^* -algebras and C^* -envelopes.

The theory of unbounded elements affiliated with a C^* -algebra was developed in [Wor1], see also [Lan]. Let \mathfrak{A} be a C^* -algebra and let T be a densely defined closed linear operator on \mathfrak{A} . Denote by $D(T) \subseteq \mathfrak{A}$ its domain¹. The adjoint operator T^* is defined as follows. For $y, z \in \mathfrak{A}$ write $y \in D(T^*)$, $T^*y = z$ if $\langle Tx, y \rangle = \langle x, z \rangle$ holds for all $x \in D(T)$. Following [Lan] we say that T is *affiliated*² with \mathfrak{A} and write $T \eta \mathfrak{A}$, if $D(T^*)$ and the range of $1 + T^*T$ are dense in \mathfrak{A} , see [Lan, Chapter 9].

Every non-degenerate *-representation of a C^* -algebra \mathfrak{A} can be continued to the set \mathfrak{A}^η of all operators affiliated with \mathfrak{A} . Namely, for every $\pi \in \text{Rep}(\mathfrak{A}, \mathcal{H})$ and $T \eta \mathfrak{A}$, there exists a closed operator $\pi(T) \eta \pi(\mathfrak{A})$ with a core $\pi(D(T))\mathcal{H}$ such that

$$\pi(T)(\pi(a)\varphi) = \pi(Ta)\varphi, \text{ for all } \varphi \in \mathcal{H}, a \in D(T).$$

Moreover, if $D_0 \subseteq D(T)$ is a core of T , then $\pi(D_0)\mathcal{H}$ is a core of $\pi(T)$.

Definition 0.1. Let \mathcal{A} be a *-algebra with a given category of *-representations $\text{Rep}\mathcal{A}$ and fixed generators a_1, \dots, a_n . We will say that a C^* -algebra \mathfrak{A} is a *C^* -envelope* of \mathcal{A} if there exist affiliated elements $A_1, \dots, A_n \eta \mathfrak{A}$ such that

$$(0.3) \quad \pi(A_i) = \overline{\rho(a_i)}, \quad i = 1, \dots, n.$$

defines an equivalence functor $\rho \mapsto \pi$ between $\text{Rep}\mathcal{A}$ and $\text{Rep}\mathfrak{A}$.

Remarks 1. If every *-representation of \mathcal{A} is bounded, then there exists the enveloping C^* -algebra $C_{env}^*(\mathcal{A})$, which is obviously a C^* -envelope of \mathcal{A} .
2. In the last definition, the isomorphism class of \mathfrak{A} depends a priori on the choice of the generators a_i and of the category $\text{Rep}\mathcal{A}$. However, we cannot provide any example, where \mathfrak{A} would depend on the generators a_i .

1. THE ORBIT METHOD

In this section we recall the orbit method developed in [SS]. Throughout the section G is a countable discrete group and \mathcal{A} is a G -graded *-algebra. We assume that the *-subalgebra $\mathcal{B} := \mathcal{A}_e$ is commutative and denote by $\widehat{\mathcal{B}}$ the set of all characters of \mathcal{B} (i.e. nontrivial *-homomorphisms $\chi : \mathcal{B} \rightarrow \mathbb{C}$). Further, we define the "positive" spectrum $\widehat{\mathcal{B}}^+ \subseteq \widehat{\mathcal{B}}$ to be the set of all characters $\chi \in \widehat{\mathcal{B}}$ which satisfy³

$$(1.1) \quad \chi(a^*a) \geq 0 \text{ for all homogeneous elements } a \in \mathcal{A}.$$

¹Recall that $\mathcal{D}(\cdot)$ is domain of a Hilbert space operator.

²In [Lan] the term *regular operator* on \mathfrak{A} is used.

³The theory developed in [SS] requires the additional condition $\chi(c^*d)\chi(d^*c) = \chi(c^*c)\chi(d^*d)$ for all $\chi \in \widehat{\mathcal{B}}^+, g \in G, c, d \in \mathcal{A}_g$, which holds automatically. It can be checked using the equation $(c^*cd^*d)^2 = (c^*cd^*d)(c^*dd^*c)$ which follows by commutativity of \mathcal{B} .

Lemma 1.1. *Assume that for every $g \in G$ there exists an element $a_g \in \mathcal{A}_g$ such that $\mathcal{A}_g = a_g \mathcal{B}$. Then $\chi \in \widehat{\mathcal{B}}$ belongs to $\widehat{\mathcal{B}}^+$ if and only if $\chi(a_g^* a_g) \geq 0$ for all $g \in G$.*

Proof. The “only if” part is clear. Assume that $\chi(a_g^* a_g) \geq 0$ for all $g \in G$. By assumption, if $c_g \in \mathcal{A}_g$, then $c_g = a_g b$ for some $b \in \mathcal{B}$. Hence

$$\chi(c_g^* c_g) = \chi(b^* a_g^* a_g b) = \chi(a_g^* a_g) \chi(b^* b) \geq 0.$$

□

The set $\widehat{\mathcal{B}}^+$ consists of those characters which satisfy (0.1), i.e. are inducible from \mathcal{B} to \mathcal{A} via p_e .

Definition 1.1. For $g \in G$ define⁴

$$(1.2) \quad \mathcal{D}_{g^{-1}} = \left\{ \chi \in \widehat{\mathcal{B}}^+ \mid \chi(a_g^* a_g) \neq 0 \text{ for some } a_g \in \mathcal{A}_g \right\}.$$

If $\chi \in \mathcal{D}_{g^{-1}}$ and $\chi(a_g^* a_g) \neq 0$ we set

$$(1.3) \quad (\alpha_g(\chi))(b) := \frac{\chi(a_g^* b a_g)}{\chi(a_g^* a_g)} \text{ for } b \in \mathcal{B}.$$

Direct computations (see [SS, Proposition 13]) show that $\alpha = (\{\alpha_g\}_{g \in G}, \{\mathcal{D}_g\}_{g \in G})$ is a well-defined partial action of G on $\widehat{\mathcal{B}}^+$. We will often write χ^g instead of $\alpha_g(\chi)$. For a character $\chi \in \widehat{\mathcal{B}}^+$ we denote by $\text{Orb}\chi \subseteq \widehat{\mathcal{B}}^+$ its orbit under the partial action of G .

Proposition 1.2 (see Proposition 16 in [SS]). *Let $\chi \in \widehat{\mathcal{B}}^+$ and $\pi = \text{Ind}\chi$ be the induced $*$ -representation. For every $g \in G$ such that $\chi \in \mathcal{D}_{g^{-1}}$ fix an element $a_g \in \mathcal{A}_g$ such that $\chi(a_g^* a_g) \neq 0$. Then there exists an orthonormal base $\{e_g \mid \chi \in \mathcal{D}_{g^{-1}}\}$ in $\mathcal{D}(\pi)$ such that for $h \in G$ and $b_h \in \mathcal{A}_h$ we have*

$$\pi(b_h)e_g = \frac{\chi(a_{hg}^* b_h a_g)}{\chi(a_{hg}^* a_{hg})^{1/2} \chi(a_g^* a_g)^{1/2}} e_{hg}, \text{ if } \chi \in \mathcal{D}_{g^{-1}h^{-1}}$$

and $\pi(b_h)e_g = 0$ otherwise. In particular, if $b \in \mathcal{B}$, we have $\pi(b)e_g = \chi^g(b)e_g$.

For an element $b \in \mathcal{B}$ introduce its “Gel’fand transform”

$$\widehat{b} : \widehat{\mathcal{B}} \rightarrow \mathbb{C}, \quad \widehat{b}(\chi) = \chi(b), \quad \chi \in \widehat{\mathcal{B}}.$$

We equip $\widehat{\mathcal{B}}$ with the weak topology defined by $\{\widehat{b} \mid b \in \mathcal{B}\}$ and the Borel structure generated by the open sets. By definition of $\widehat{\mathcal{B}}^+$ it is a closed subset of $\widehat{\mathcal{B}}$. It can be checked, that the partial action of G is topological. That is, \mathcal{D}_g , $g \in G$ are open sets, and $\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g$ are homeomorphisms. Since G is countable and the one-point sets are closed, the G -orbits are Borel subsets of $\widehat{\mathcal{B}}^+$.

Definition 1.2. A closed $*$ -representation π of \mathcal{A} is called *well-behaved* if:

- (i) the restriction $\text{Res}_{\mathcal{B}}\pi$ of π to \mathcal{B} is integrable and there exists a spectral measure E_π on $\widehat{\mathcal{B}}^+$ such that

$$\overline{\pi(b)} = \int_{\widehat{\mathcal{B}}^+} \widehat{b}(\chi) dE_\pi(\chi) \text{ for } b \in \mathcal{B}.$$

⁴In [SS] the notation $\alpha_g : \mathcal{D}_g \rightarrow \mathcal{D}_{g^{-1}}$ was used.

(ii) For all $a_g \in \mathcal{A}_g, g \in G$, and all Borel subsets $\Delta \subseteq \widehat{\mathcal{B}}^+$, we have

$$(1.4) \quad \pi(a_g)E_\pi(\Delta) \supseteq E_\pi(\alpha_g(\Delta \cap \mathcal{D}_{g^{-1}}))\pi(a_g).$$

A well-behaved *-representation π is *associated with an orbit* $\text{Orb}\chi$ if E_π is supported on the set $\text{Orb}\chi$. Denote by $\text{Rep}\mathcal{A}$ the category of well-behaved *-representations.

By [SS, Proposition 17], relation (1.4) can be replaced with

$$(1.5) \quad u_g \int f(t) dE_\pi(t) \subseteq \int_{\mathcal{D}_g} f(\alpha_{g^{-1}}(t)) dE_\pi(t) \cdot u_g.$$

where u_g is the partial isometry in the polar decomposition $\overline{\pi(a_g)} = u_g c_g$, and f is any measurable function on $\widehat{\mathcal{B}}^+$. If f is bounded, then “ \subseteq ” becomes an equality.

In the next proposition we collect basic properties of well-behaved *-representations. For the proof see Propositions 18, 29 and Theorem 7 in [SS].

Proposition 1.3. (i) *Every bounded *-representation is well-behaved.*

(ii) *If the partial action of G on $\widehat{\mathcal{B}}^+$ possesses a measurable countably separated section, then every irreducible well-behaved *-representation is associated with an orbit.*

(iii) *Condition (i) in Definition 1.2 holds automatically if \mathcal{B} is countably generated, and the restriction of π on \mathcal{B} is integrable, that is $\pi(a)$ is normal for all $b \in \mathcal{B}$.*

A measurable set Γ is *countably separated* if and only if there exist Borel sets $B_k, k \in \mathbb{N}, \Gamma \subseteq \bigcup_{k \in \mathbb{N}} B_k$ such that for arbitrary $x, y \in \Gamma, x \neq y$, we have $x \in B_{k_0}, y \notin B_{k_0}$ for some $k_0 \in \mathbb{N}$. A subset Γ containing exactly one point from each orbit is called a *section* of a partial dynamical system.

Recall, that for a subgroup $H \subseteq G, \mathcal{A}_H = \bigoplus_{g \in H} \mathcal{A}_g$ is a *-subalgebra of \mathcal{A} .

Theorem 1.4 (See Theorem 5 in [SS]). *Let $\chi \in \widehat{\mathcal{B}}^+$ be a character and let $H = \text{St}\chi$ be its stabilizer group. Then the map*

$$\rho \mapsto \text{Ind}_{\mathcal{A}_H \uparrow \mathcal{A}}(\rho) = \pi$$

*is a bijection from the set of unitary equivalence classes of inducible *-representations ρ of \mathcal{A}_H for which*

$$(1.6) \quad \text{Res}_{\mathcal{B}} \rho \text{ corresponds to a multiple of the character } \chi$$

*onto the set of unitary equivalence classes of well-behaved *-representations π of \mathcal{A} associated with $\text{Orb}\chi$. A *-representation ρ satisfying (1.6) is bounded and inducible. Moreover, π is irreducible if and only if ρ is irreducible.*

The last theorem suggests the following algorithm for description of all irreducible well-behaved *-representations of \mathcal{A} :

- determine $\widehat{\mathcal{B}}, \widehat{\mathcal{B}}^+$, the partial action of G on $\widehat{\mathcal{B}}^+$ and a section $\Gamma \subseteq \widehat{\mathcal{B}}^+$,
- for each $\chi \in \Gamma$
 - if the stabilizer $\text{St}\chi$ is trivial, compute $\text{Ind}\chi$,
 - otherwise find all irreducible representations ρ of $\mathcal{A}_{\text{St}\chi}$ satisfying (1.6) and compute $\text{Ind}\rho$.

If the Proposition 1.3, (ii) applies, then we obtain all irreducible well-behaved representations of \mathcal{A} .

2. THE q -OSCILLATOR ALGEBRA

By the *quantum harmonic oscillator* (q -oscillator) we mean the following relation

$$(2.1) \quad aa^* = \mathbf{1} + qa^*a, \quad q > 0.$$

In this section, we use the notation of the q -calculus $[[k]]_q = 1 + q + \dots + q^{k-1}$. Further, we put

$$F(t) := 1 + qt.$$

Clearly $F([[k]]_q) = [[k+1]]_q$, $k \in \mathbb{N}_0$.

In [CGP] the authors have obtained the following representations of (2.1) by Hilbert space operators:

- For every $q > 0$ the Fock representation π_F acting on the orthonormal base $\{\mathbf{e}_k\}_{k \in \mathbb{N}_0}$ as

$$(2.2) \quad \pi_F(a)\mathbf{e}_k = [[k]]_q^{1/2}\mathbf{e}_{k-1}, \quad \pi_F(a^*)\mathbf{e}_k = [[k+1]]_q^{1/2}\mathbf{e}_{k+1}, \quad \text{where } \mathbf{e}_{-1} := 0.$$

- For $q \in (0, 1)$ the series of unbounded $*$ -representations $\pi_\gamma, \gamma \in (0, 1]$ acting on the orthonormal base $\{\mathbf{e}_k\}_{k \in \mathbb{Z}}$ as

$$(2.3) \quad \pi_\gamma(a)\mathbf{e}_k = \left(\frac{1+q^{\gamma+k}}{1-q}\right)^{1/2}\mathbf{e}_{k+1}, \quad \pi_\gamma(a^*)\mathbf{e}_k = \left(\frac{1+q^{\gamma+k+1}}{1-q}\right)^{1/2}\mathbf{e}_{k-1}.$$

- For $q \in (0, 1)$ the series of one-dimensional $*$ -representations

$$(2.4) \quad \pi_\varphi(a) = e^{i\varphi}(1-q)^{-1/2}, \quad \pi_\varphi(a^*) = e^{-i\varphi}(1-q)^{-1/2}, \quad \varphi \in [0, 2\pi).$$

Using the orbit method described in the previous section, we classify all irreducible well-behaved $*$ -representations of the q -oscillator algebra

$$\mathcal{A} = \mathbb{C}\langle a, a^* \mid aa^* = qa^*a + \mathbf{1} \rangle, \quad q > 0.$$

We will see that the formulas for the irreducible well-behaved $*$ -representations of \mathcal{A} coincide with (2.2)–(2.4).

We now introduce the ingredients needed for the orbit method. Define the \mathbb{Z} -grading on \mathcal{A} by setting $a \in \mathcal{A}_1$, $a^* \in \mathcal{A}_{-1}$ and put $\mathcal{B} := \mathcal{A}_0$. It is easily checked that $\mathcal{B} = \mathbb{C}[N]$, where $N = a^*a$, and $\mathcal{A}_n = a^n\mathcal{B}$, $\mathcal{A}_{-n} = a^{*n}\mathcal{B}$ for every $n \in \mathbb{N}$. Using induction on $k \in \mathbb{N}$ we obtain the relations

$$(2.5) \quad \begin{aligned} a^k a^{*k} &= \prod_{j=1}^k (q^j N + [[j]]_q \mathbf{1}), \quad k \in \mathbb{N}. \\ a^{*k} a^k &= \prod_{j=0}^{k-1} (q^{-j} N + [[-j]]_q \mathbf{1}), \quad k \in \mathbb{N}. \end{aligned}$$

Since $\mathcal{B} = \mathbb{C}[N]$, every character on \mathcal{B} is of the form $\chi_t(N) = t \in \mathbb{R}$. In what follows we identify the space of all characters $\widehat{\mathcal{B}}$ with \mathbb{R} .

Proposition 2.1. (i) $\widehat{\mathcal{B}}^+ = \{[[k]]_q \mid k \in \mathbb{N}_0\}$ for $q \geq 1$,

$$\widehat{\mathcal{B}}^+ = \{[[k]]_q \mid k \in \mathbb{N}_0\} \cup [1/(1-q), +\infty) \text{ for } q \in (0, 1).$$

(ii) The partial action $\alpha = (\{\mathcal{D}_n\}_{n \in \mathbb{Z}}, \{\alpha_n\}_{n \in \mathbb{Z}})$ is given as follows.

$$\mathcal{D}_{-n} = \{[[k]]_q \mid k \geq n\} \text{ if } q \geq 1,$$

$$\mathcal{D}_{-n} = \{[[k]]_q \mid k \geq n\} \cup [1/(1-q), \infty) \text{ if } q \in (0, 1).$$

If $\chi_t \in \mathcal{D}_{-n}$, then $\chi_t^n = \chi_{F^{-n}(t)}$. In particular, $\chi_{[[k]]_q}^n = \chi_{[[k-n]]_q}$ for $n \leq k$.

Proof. (i) By Lemma 1.1 a character $\chi \in \widehat{\mathcal{B}}$ belongs to $\widehat{\mathcal{B}}^+$ if and only if $\chi(a^k a^{*k}) \geq 0$, $\chi(a^{*k} a^k) \geq 0$ for all $k \in \mathbb{N}$. Further, (2.5) implies that $a^n a^{*n} = \sum_{j=0}^n \alpha_j N^j$ for some $\alpha_j \geq 0$. Hence $\chi_t \in \widehat{\mathcal{B}}^+$ if and only if $\chi(a^{*k} a^k) = \chi\left(\prod_{j=0}^{k-1} (q^{-j} N + [[-j]]_q \mathbf{1})\right) \geq 0$ for all $k \in \mathbb{N}$. The last system of inequalities is equivalent to

$$(2.6) \quad \prod_{j=0}^k (t - [[j]]_q) \geq 0 \text{ for all } k \in \mathbb{N}_0.$$

Consider first $q \geq 1$. Then $[[k]]_q \rightarrow \infty$, $k \rightarrow \infty$, and (2.6) is satisfied if and only if $t = [[k]]_q$ for some $k \in \mathbb{N}_0$. If $q \in (0, 1)$, then $[[k]]_q \rightarrow \frac{1}{1-q}$, $k \rightarrow \infty$, and every $t \geq \frac{1}{1-q}$ satisfies (2.6). For $t \in \left[0, \frac{1}{1-q}\right)$, (2.6) holds if and only if $t = [[k]]_q$ for some $k \in \mathbb{N}_0$.

(ii) One can verify by induction on $n \in \mathbb{N}$ that $F^n(t) = q^n t + [[n]]_q$ for all $n \in \mathbb{Z}$. Using (2.5), we obtain

$$\chi_t^n(N) = \frac{\chi_t(a^{*n} N a^n)}{\chi_t(a^{*n} a^n)} = \chi_t(q^{-n} N + [[-n]]_q) = \chi_{F^{-n}(t)}(N),$$

for $\chi_t \in \mathcal{D}_{-n}$, $n \in \mathbb{N}$, and

$$\chi_t^{-n}(N) = \frac{\chi_t(a^n N a^{*n})}{\chi_t(a^n a^{*n})} = q^{-1} \chi_t(q^{n+1} N + [[n+1]]_q) - q^{-1} = \chi_{F^n(t)}(N).$$

for $\chi_t \in \mathcal{D}_n$, $n \in \mathbb{N}$. Inequalities (2.6) imply that for $q > 0$ and $t = [[k]]_q$, we have $\chi_t \in \mathcal{D}_{-n}$ if and only if $n \leq k$. In case $q \in (0, 1)$ and $t \geq \frac{1}{1-q}$, we have $\chi_t \in \mathcal{D}_{-n}$ for all $n \in \mathbb{Z}$. \square

Using Proposition 2.1, we conclude that the stabilizer St_{χ_t} of $\chi_t \in \widehat{\mathcal{B}}^+$ is trivial except for the case $t = 1/(1-q)$, where the stabilizer is \mathbb{Z} . Define the subset $\Gamma \subseteq \widehat{\mathcal{B}}^+$ as

$$\Gamma = \{0\} \cup \left\{ \frac{1}{1-q} \right\} \cup \left\{ \frac{1+q^\gamma}{1-q} \mid \gamma \in (0, 1] \right\}, \text{ if } q \in (0, 1),$$

$$\Gamma = \{0\}, \text{ if } q \geq 1.$$

Direct computations using Proposition 2.1 show that each orbit under the partial action of \mathbb{Z} on $\widehat{\mathcal{B}}^+$ intersects Γ in exactly one point, i.e. Γ is a section of the partial action. The topology on $\widehat{\mathcal{B}}^+$ is induced from the standard topology on \mathbb{R} . Hence Γ is countably separated and measurable. By Proposition 1.3(ii) every irreducible well-behaved *-representation of \mathcal{A} is associated to some $\text{Orb}\chi$, $\chi \in \Gamma$. For $\chi \in \Gamma$ we consider three cases.

- (i) Case $\chi = \chi_0$. Since the stabilizer of χ is trivial, the only irreducible well-behaved *-representation associated to $\text{Orb}\chi$ is (up to unitary equivalence) $\pi_F := \text{Ind}\chi$. Using Proposition 1.2 and relations (2.5), we calculate the action of π_F on the orthonormal basis $\{e_{-n}\}_{n \in \mathbb{N}_0}$ of the representation

space \mathcal{H}_{π_F} .

$$\begin{aligned}\pi_F(a)e_{-n} &= \frac{\chi(a^{n-1}aa^{*n})}{\chi(a^{n-1}a^{*(n-1)})^{1/2}\chi(a^n a^{*n})^{1/2}}e_{-n+1} = \frac{\chi(a^n a^{*n})^{1/2}}{\chi(a^{n-1}a^{*(n-1)})^{1/2}}e_{-n+1} \\ &= q^{n/2}(\chi(N) - [[-n]]_q)^{1/2}e_{-n+1} = [[n]]_q^{1/2}e_{-n+1}, \\ \pi_F(a^*)e_{-n} &= \frac{\chi_0(a^{n+1}a^*a^{*n})}{\chi_0(a^{n+1}a^{*(n+1)})^{1/2}\chi_0(a^n a^{*n})^{1/2}}e_{-n-1} = \frac{\chi_0(a^{n+1}a^{*(n+1)})^{1/2}}{\chi_0(a^n a^{*n})^{1/2}}e_{-n-1} \\ &= q^{(n+1)/2}(\chi(N) - [[-n-1]]_q)^{1/2}e_{-n-1} = [[n+1]]_q^{1/2}e_{-n-1},\end{aligned}$$

where $e_1 := 0$ and $n \in \mathbb{N}_0$. It exists for any $q > 0$ and is bounded if and only if $q \in (0, 1)$.

- (ii) Case $\chi = \chi_{\frac{1+q^\gamma}{1-q}}$, $\gamma \in (0, 1]$. The stabilizer of χ is again trivial, thus $\pi_\gamma := \text{Ind} \chi_{\frac{1+q^\gamma}{1-q}}$ is the only irreducible well-behaved $*$ -representation associated to $\text{Orb} \chi$. We calculate the action of $\pi_\gamma(a)$ respectively $\pi_\gamma(a^*)$ using Proposition 1.2 and relations (2.5). For $n \in \mathbb{Z}$ we have

$$\begin{aligned}\pi_\gamma(a)e_n &= \frac{\chi(a^{*(n+1)}aa^n)}{\chi(a^{*(n+1)}a^{n+1})^{1/2}\chi(a^{*n}a^n)^{1/2}}e_{n+1} = \frac{\chi(a^{*(n+1)}a^{n+1})^{1/2}}{\chi(a^{*n}a^n)^{1/2}}e_{n+1} \\ &= \left(q^{-n}\frac{1+q^\gamma}{1-q} + \frac{1-q^{-n}}{1-q}\right)^{1/2}e_{n+1} = \left(\frac{1+q^{\gamma-n}}{1-q}\right)^{1/2}e_{n+1}.\end{aligned}$$

In the same way we obtain

$$\pi_\gamma(a^*)e_n = \left(\frac{1+q^{\gamma-n+1}}{1-q}\right)^{1/2}e_{n-1}, \text{ for } n \in \mathbb{Z}.$$

Note that π_γ is not bounded for every $\gamma \in (0, 1]$.

- (iii) Case $\chi = \chi_{\frac{1}{1-q}}$. The stabilizer group H of $\chi_{\frac{1}{1-q}}$ is \mathbb{Z} . Let ρ be an irreducible $*$ -representation of \mathcal{A} satisfying (1.6). Since $\chi(aa^* - a^*a) = 0$, we have $\rho(a)\rho(a^*) = \rho(a^*)\rho(a)$. By Schur's Lemma ρ is one-dimensional. For $\lambda \in \mathbb{C}$ such that $\lambda = \rho(a)$ we get $|\lambda|^2 = \rho(aa^*) = \rho(1 + qa^*a) = 1 + q|\lambda|^2$. Hence $\rho = \rho_\varphi$, for some $\varphi \in [0, 2\pi)$, where $\rho_\varphi(a) = e^{i\varphi}(1-q)^{-1/2}$. Since $\mathcal{A}_H = \mathcal{A}$, $\pi_\varphi := \text{Ind}_{\mathcal{A}_H \uparrow \mathcal{A}} \rho_\varphi$ is equivalent to ρ_φ and we have

$$\pi_\varphi(a) = e^{i\varphi}(1-q)^{-1/2}, \quad \pi_\varphi(a^*) = e^{-i\varphi}(1-q)^{-1/2}, \quad \varphi \in [0, 2\pi).$$

By Theorem 1.4 these are all (up to unitary equivalence) irreducible well-behaved $*$ -representations of \mathcal{A} . Moreover, putting $e_k := e_{-k}$, $k \in \mathbb{Z}$, we see that the above formulas coincide with (2.2), (2.3) and (2.4) respectively. We have proved the following

Theorem 2.2. *Every irreducible well-behaved $*$ -representation of the q -oscillator algebra, $q > 0$, is induced from a one-dimensional $*$ -representation.*

2.1. Existence of bad $*$ -representations. In this subsection we prove the existence of a $*$ -representation π of \mathcal{A} which is not well-behaved and which cannot be continued to a well-behaved representation in a possibly larger Hilbert space. The idea is similar to the proof of [S1, Theorem 4.1].

Lemma 2.3. *The polynomial*

$$p := (N-1)(N-(1+q)) \in \mathbb{C}[N]$$

is positive in every well-behaved $$ -representation of \mathcal{A} and $p \notin \sum \mathcal{A}^2$.*

Proof. We first show that every element of $\sum \mathcal{A}^2 \cap \mathcal{B}$ is of the form

$$(2.7) \quad \sum_{k=0}^n a^{*k} a^k \cdot p_k^* p_k, \text{ where } p_k \in \mathbb{C}[N], \ n \in \mathbb{N}.$$

Indeed, an element $b \in \mathcal{B}$ belongs to $\sum \mathcal{A}^2$ if and only if $b = \sum b_j^* b_j$, where $b_j \in \mathcal{A}_{k_j}$, $k_j \in \mathbb{Z}$. Since $\mathcal{A}_k = a^k \cdot \mathcal{B}$ for every $k \in \mathbb{Z}$ (here $a^{-k} = a^{*k}$ for $k > 0$), we obtain $b = \sum a^{*k} a^k \cdot s_k$, where $s_k \in \sum \mathcal{B}^2$. It is a well-known fact, that every positive polynomial in $\mathbb{C}[N]$ is a single square $r^* r$. Hence $s_k = p_k^* p_k$, $p_k \in \mathbb{C}[N]$, for $k \in \mathbb{Z}$. Furthermore, relations (2.5) imply $a^n a^{*n} \in \sum \mathcal{B}^2 + a^* a \sum \mathcal{B}^2$, which proves (2.7).

Let π be a well-behaved *-representation of \mathcal{A} with associated spectral measure E_π . Since $\text{supp } E_\pi \subseteq \widehat{\mathcal{B}}^+$ and $p \geq 0$ on $\widehat{\mathcal{B}}^+$, we have $\pi(p) = \int_{\widehat{\mathcal{B}}^+} p(\lambda) dE_\pi(\lambda) \geq 0$.

Assume to the contrary that $p \in \sum \mathcal{A}^2$. Since the degree of $p(N)$ in $\mathbb{C}[N]$ is 2, we get by (2.7)

$$p = f^* f + a^* a \cdot g^* g + a^{*2} a^2 \cdot h^* h = f^* f + N g^* g + N(N-1) h^* h$$

for some polynomials $f, g, h \in \mathbb{C}[N]$, where $\deg f \leq 1$, $\deg g = 0$ and $\deg h = 0$, that is, g and h are constant. Setting $N := 1$, we obtain $|f(1)|^2 + |g|^2 = 0$, i.e. $g = 0$, $f(1) = 0$. Setting $N := 1 + q$, we get $|f(1+q)|^2 + q(1+q)|h|^2 = 0$ which implies $h = f(1+q) = 0$. Since $\deg f \leq 1$, $f \equiv 0$, i.e. $p \equiv 0$, a contradiction. \square

For the prove of the next theorem we will need the following technical result, see [S2, Lemma 2].

Lemma 2.4. *Let \mathcal{A} be a unital *-algebra which has a faithful *-representation π (that is, $\pi(a) = 0$ implies that $a = 0$) and is a union of a sequence of finite dimensional subspaces E_n , $n \in \mathbb{N}$. Assume that for each $n \in \mathbb{N}$ there exists a number $k_n \in \mathbb{N}$ such that the following is satisfied: If $a \in \sum \mathcal{A}^2$ is in E_n , then we can write a as a finite sum $\sum a_j^* a_j$ such that all a_j are in E_{k_n} . Then the cone $\sum \mathcal{A}^2$ is closed in \mathcal{A} with respect to the finest locally convex topology on \mathcal{A} .*

Theorem 2.5. *There exists a *-representation π of the q -oscillator algebra \mathcal{A} which cannot be extended to a well-behaved representation in a possibly larger Hilbert space.*

Proof. Since $p \notin \sum \mathcal{A}^2$ and $\sum \mathcal{A}^2$ is closed by Lemma 2.4, there exists a linear functional $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ such that $\varphi(\sum \mathcal{A}^2) \geq 0$ and $\varphi(p) < 0$ by the Hahn-Banach Theorem. Let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be its GNS-construction (see [S3, Section 8.6.]). Assume to the contrary, that π_φ has a well-behaved extension, say π . Then $\langle \pi(p) \xi_\varphi, \xi_\varphi \rangle = \langle \pi_\varphi(p) \xi_\varphi, \xi_\varphi \rangle = \varphi(p) < 0$. On the other hand, $\pi(p) \geq 0$ by Lemma 2.3, a contradiction. \square

2.2. C*-envelope of the q -oscillator algebra. In this subsection we show that \mathcal{A} , considered with the category $\text{Rep } \mathcal{A}$ and generators a, a^* has a C*-envelope \mathfrak{A} in the sense of Definition 0.1. For let $(C_0(\widehat{\mathcal{B}}^+), \mathbb{Z}, \beta)$ be the C*-p.d.s. dual to $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$. More precisely, define the partial action $\beta = (\{I_k\}_{k \in \mathbb{Z}}, \{\beta_k\}_{k \in \mathbb{Z}})$ on $C_0(\widehat{\mathcal{B}}^+)$ by setting $I_k := C_0(\mathcal{D}_k)$ and

$$(\beta_k(f))(t) := f(\alpha_{-k}(t)) = f(F^k(t)), \text{ for } f \in I_{-k}, \ t \in \mathcal{D}_k.$$

Proposition 2.1 implies that the C^* -p.d.s. $(C_0(\widehat{\mathcal{B}}^+), \mathbb{Z}, \beta)$ is defined by the partial automorphism $\Theta = (\theta, I, J)$, where

$$I = I_{-1}, \quad J = I_1 = C_0(\widehat{\mathcal{B}}^+), \quad (\theta(f))(t) = (\beta_1(f))(t) = f(1 + qt), \quad f \in I_{-1}.$$

We define

$$\mathfrak{A} := C^*(C_0(\widehat{\mathcal{B}}^+), \Theta) = C_0(\widehat{\mathcal{B}}^+) \times_{\beta} \mathbb{Z}.$$

Theorem 2.6. *Consider the q -oscillator algebra \mathcal{A} with generators a, a^* and the category of well-behaved representations $\text{Rep}\mathcal{A}$. Then \mathfrak{A} is a C^* -envelope of \mathcal{A} .*

Proof. Let \mathfrak{A}_0 be the linear hull of

$$\{f \otimes k \in \mathfrak{A} \mid k \in \mathbb{Z}, \text{supp} f \subseteq \mathcal{D}_k \text{ is compact}\}.$$

\mathfrak{A}_0 is obviously dense in \mathfrak{A} . For $f \otimes k \in \mathfrak{A}_0$ define

$$A(f(t) \otimes k) = \sqrt{1 + qt} f(1 + qt) \otimes (k + 1),$$

and

$$A^*(f(t) \otimes k) = \sqrt{t} f(q^{-1}t - q^{-1}) \otimes (k - 1)$$

Then A and A^* are densely defined linear operators on \mathfrak{A} and their closures, denoted again by A and A^* , are adjoint to each other. For $f \otimes k \in \mathfrak{A}_0$ we have

$$A^*A(f(t) \otimes k) = A^*(\sqrt{\alpha_{-1}(t)} f(\alpha_{-1}(t)) \otimes (k + 1)) = tf(t) \otimes k.$$

The last equation shows that the range of $I + A^*A$ is dense in $\mathfrak{A}_0 \subseteq \mathfrak{A}$, so that A is affiliated with \mathfrak{A} . By [Wor1, Theorem 1.4.] the adjoint A^* is also affiliated with \mathfrak{A} . We show the correspondence (0.3) between the generators $a, a^* \in \mathcal{A}$ and affiliated elements $A, A^* \eta \mathfrak{A}$.

By [Ex, Theorem 5.6] every $*$ -representation of \mathfrak{A} is given by a covariant representation $\pi \times u$ of $(C_0(\widehat{\mathcal{B}}^+), \Theta)$. Here $\pi : C_0(\widehat{\mathcal{B}}^+) \rightarrow B(\mathcal{H}_{\pi})$ is a $*$ -representation of $C_0(\widehat{\mathcal{B}}^+)$ and u is a partial isometry on \mathcal{H}_{π} satisfying $\pi(\theta(b)) = u\pi(b)u^*$ for every $b \in I$. By the spectral theory of commutative C^* -algebras, there exists a unique spectral measure E_{π} on $\widehat{\mathcal{B}}^+$ such that

$$\pi(f) = \int_{\widehat{\mathcal{B}}^+} f(t) dE_{\pi}(t), \quad f \in C_0(\widehat{\mathcal{B}}^+).$$

By definition of θ for $f \in I_{-1} = C_0(\mathcal{D}_{-1})$ we have

$$u \left(\int f dE_{\pi} \right) u^* = u\pi(f)u^* = \pi(\theta(f)) = \int f(1 + qt) dE_{\pi}(t).$$

Multiplying the latter by u from the right and remembering that the initial space of u is $\overline{\pi(I_{-1})\mathcal{H}_{\pi}} = E_{\pi}(\mathcal{D}_{-1})\mathcal{H}_{\pi}$ we get

$$(2.8) \quad u \int f dE_{\pi} = \int f(1 + qt) dE_{\pi}(t) \cdot u, \quad \text{for } f \in I_{-1}.$$

The extension of $(\pi \times u)$ to A and A^* is given by

$$(2.9) \quad (\pi \times u)(A) = u \int \sqrt{t} dE_{\pi}, \quad (\pi \times u)(A^*) = \int \sqrt{t} dE_{\pi}(t) \cdot u^*$$

Indeed, for every $f \otimes k \in \mathfrak{A}_0$ we have

$$\begin{aligned} u \int \sqrt{t} dE_\pi \cdot ((\pi \times u)(f \otimes k)) &= u \int \sqrt{t} dE_\pi \cdot \int f dE_\pi \cdot u^k = \\ &= u \int \sqrt{t} f(t) dE_\pi \cdot u^k = \int \sqrt{1+qt} f(1+qt) dE_\pi \cdot u^{k+1} = (\pi \times u)(A(f \otimes k)). \end{aligned}$$

Since $\mathfrak{A}_0 \subseteq \mathfrak{A}$ is a core of A , $\pi(\mathfrak{A}_0)\mathcal{H}_\pi$ is a core of A , and we get the first part of (2.9). The second part follows from $(\pi \times u)(A^*) = ((\pi \times u)(A))^*$.

Let ρ be a well-behaved *-representation of \mathcal{A} , and E_ρ be the corresponding spectral measure on $\widehat{\mathcal{B}}^+ \subseteq \mathbb{R}_+$. Further, let $\overline{\rho(a)} = u_1 c_1$ be the polar decomposition of $\overline{\rho(a)}$. Since a^*a is the generator of \mathcal{B} , E_ρ coincides with the spectral measure of $\overline{\rho(a^*a)} = \rho(a)^* \rho(a)$. Hence

$$(2.10) \quad \overline{\rho(a)} = u_1 \int \sqrt{t} dE_\rho, \text{ and } \overline{\rho(a^*)} = \int \sqrt{t} dE_\rho \cdot u_1^*.$$

Since $\ker u_1 = \ker c_1$, the initial space of u_1 is the range of $E_\rho(\widehat{\mathcal{B}}^+ \setminus \{0\}) = E_\rho(\mathcal{D}_{-1})$. Further, we have $u_1 c_1^2 u_1^* = 1 + q c_1^2$, which implies that $\ker u_1^*$ is trivial, so that the final space of u_1 is $E_\rho(\widehat{\mathcal{B}}^+) = E_\rho(\mathcal{D}_1)$. Applying (1.5) to $f \in C_0(\mathcal{D}_{-1})$ we obtain

$$\begin{aligned} u_1 \rho(f) u_1^* &= u_1 \int f(t) dE_\rho(t) \cdot u_1^* = \int f(\alpha_{-1}(t)) dE_\rho(t) \cdot u_1^* u_1 = \\ &= \int f(1+qt) dE_\rho(t) = \rho(\theta(f)), \end{aligned}$$

i.e. $(\rho_{\mathcal{B}} \times u_1)$, where $\rho_{\mathcal{B}}$ is the restriction of ρ to \mathcal{B} , defines a covariant representation of $(C_0(\widehat{\mathcal{B}}^+), \Theta)$.

The correspondence (0.3) between π and ρ follows now by comparing (2.9) with (2.10). \square

Remark. In [Wor2, Section 3] the author shows that the operators p, q of the Weyl algebra W_1 generate a C^* -algebra \mathfrak{A} in the sense of the Definition 3.1 therein, and that \mathfrak{A} is the algebra of compact operators. It corresponds to the fact that the C^* -envelope of q -CCR with $q = 1$ is isomorphic to the partial crossed product $C_0(\mathbb{N}_0) \times_\alpha \mathbb{Z} \simeq K(l^2(\mathbb{N}_0))$.

3. THE PODLEŚ SPHERE

In this section we investigate *-representations of the Podleś' sphere $\mathcal{O}(S_{qr}^2)$. We consider only the case $q \in (0, 1)$, $r \in (0, \infty)$. The cases $r = 0$, $r = \infty$ can be treated similarly. Recall [Pd] that $\mathcal{A} := \mathcal{O}(S_{qr}^2)$ is the unital *-algebra generated by $a = a^*, b, b^*$ and defining relations

$$(3.1) \quad ab = q^{-2}ba, \quad ab^* = q^2b^*a, \quad b^*b = a - a^2 + r\mathbf{1}, \quad bb^* = q^2a - q^4a^2 + r\mathbf{1}.$$

The defining relations imply that every *-representation of \mathcal{A} is bounded and hence well-behaved by Proposition 1.3 (i). In [Pd] the following irreducible *-representations of \mathcal{A} were obtained.

- Two infinite-dimensional $*$ -representations π_{\pm} which act on an orthonormal base $\{\mathbf{e}_k\}_{k \in \mathbb{N}_0}$ of the representation space \mathcal{H}_{\pm} by

$$\begin{aligned}\pi_{\pm}(a)\mathbf{e}_k &= q^{2k}\lambda_{\pm}\mathbf{e}_k, \quad \pi_{\pm}(b)\mathbf{e}_k = (q^{2k}\lambda_{\pm} - (q^{2k}\lambda_{\pm})^2 + r)^{1/2}\mathbf{e}_{k-1}, \\ \pi_{\pm}(b^*)\mathbf{e}_k &= (q^{2(k+1)}\lambda_{\pm} - (q^{2(k+1)}\lambda_{\pm})^2 + r)^{1/2}\mathbf{e}_{k+1}, \quad \mathbf{e}_{-1} := 0,\end{aligned}$$

where $\lambda_{\pm} := \frac{1}{2} \pm (r + \frac{1}{4})^{1/2}$.

- The series of one-dimensional $*$ -representations π_{φ} , $\varphi \in [0, 2\pi)$,

$$\pi_{\varphi}(a) = 0, \quad \pi_{\varphi}(b) = e^{i\varphi}r^{1/2}, \quad \pi_{\varphi}(b^*) = e^{-i\varphi}r^{1/2}.$$

Using (3.1) and induction on $n \in \mathbb{N}$ we obtain the following relations

$$(3.2) \quad \begin{aligned}ab^n &= q^{-2n}b^na, \quad ab^{*n} = q^{2n}b^{*n}a. \\ b^{*n}b^n &= \prod_{j=1}^n (q^{-2(j-1)}a - q^{-4(j-1)}a^2 + r), \quad b^nb^{*n} = \prod_{j=1}^n (q^{2j}a - q^{4j}a^2 + r).\end{aligned}$$

Define a \mathbb{Z} -grading on \mathcal{A} by setting $a \in \mathcal{A}_0$, $b \in \mathcal{A}_1$ and $b^* \in \mathcal{A}_{-1}$. Using the defining relations one easily derives

$$\mathcal{B} := \mathcal{A}_0 = \text{Lin}\{a^lb^{*m}b^m \mid l, m \in \mathbb{N}_0\}, \quad \mathcal{A}_n := b^n\mathcal{B}, \quad \mathcal{A}_{-n} := b^{*n}\mathcal{B},$$

where $n \in \mathbb{N}_0$. Further, relations (3.2) imply that $\mathcal{B} = \mathbb{C}[a]$, hence $\widehat{\mathcal{B}} = \{\chi_t \mid t \in \mathbb{R}\}$ where $\chi_t(a) = t$. As in the previous section we identify $\widehat{\mathcal{B}}$ with \mathbb{R} .

Proposition 3.1. (i) $\widehat{\mathcal{B}}^+ = \{\chi_{m,+}\}_{m \in \mathbb{N}_0} \cup \{\chi_{m,-}\}_{m \in \mathbb{N}_0} \cup \{\chi_{\infty}\}$,
where $\chi_{m,\pm}$ denotes χ_t , $t = q^{2m}\lambda_{\pm}$ and χ_{∞} denotes χ_t , $t = 0$.
(ii) The partial action $\alpha = (\{\mathcal{D}_n\}_{n \in \mathbb{Z}}, \{\alpha_n\}_{n \in \mathbb{Z}})$ is given as follows:

$$\mathcal{D}_{-n} = \{\chi_{m,\pm} \mid m \geq n\} \cup \{\chi_{\infty}\}, \quad \text{and } \chi_{m,\pm}^n = \chi_{m-n,\pm}, \quad \chi_{\infty}^n = \chi_{\infty}.$$

Proof. (i) Lemma 1.1 and relations (3.2) imply that χ_t , $t \in \mathbb{R}$ belongs to $\widehat{\mathcal{B}}^+$ if and only if the following inequalities are satisfied for all $n \in \mathbb{N}$:

$$(3.3) \quad \begin{aligned}\chi_t(b^{*n}b^n) &= \prod_{k=0}^{n-1} (q^{-2k}t - q^{-4k}t^2 + r) \geq 0, \\ \chi_t(b^nb^{*n}) &= \prod_{k=1}^n (q^{2k}t - q^{4k}t^2 + r) \geq 0.\end{aligned}$$

Assume $q^{-2k}t - q^{-4k}t^2 + r > 0$ for all $k \in \mathbb{N}_0$. Since $q^{-2k} \rightarrow +\infty$, $k \rightarrow +\infty$, it is possible only if $t = 0$, i.e. $\chi_t = \chi_{\infty}$. If $t \neq 0$ we get $q^{-2k}t - q^{-4k}t^2 + r = 0$ for some $k \in \mathbb{N}_0$, whence

$$t = \frac{-q^{-2k} \pm \sqrt{q^{-4k} + 4q^{-4k}r}}{-2q^{-4k}} = q^{2k}\lambda_{\mp}.$$

One can easily check that every $t = q^{2k}\lambda_{\pm}$, $k \in \mathbb{N}_0$, satisfies (3.3).

(ii) Relations (3.3) imply that $\mathcal{D}_{-n} = \{\chi_{m,\pm} \mid m \geq n\} \cup \{\chi_{\infty}\}$. Assume that $\chi_{m,\pm} \in \mathcal{D}_{-n}$, where $n \in \mathbb{N}_0$. Using relations (3.2) we obtain

$$\chi_{m,\pm}^n(a) = \frac{\chi_{m,\pm}(b^{*n}ab^n)}{\chi_{m,\pm}(b^{*n}b^n)} = \frac{\chi_{m,\pm}(b^{*n}b^n)\chi_{m,\pm}(q^{-2n}a)}{\chi_{m,\pm}(b^{*n}b^n)} = \chi_{m-n,\pm}(a).$$

For χ_∞ we have $\chi_\infty(b^{*n}b^n) = \chi_\infty(b^n b^{*n}) = r^n \neq 0$ for all $n \in \mathbb{Z}$ by equations (3.3). Hence $\chi_\infty \in \mathcal{D}_n$ for all $n \in \mathbb{Z}$ and $\chi_\infty^n(a) = 0$. \square

Let Γ be the subset $\{\chi_{0,+}, \chi_{0,-}, \chi_\infty\} \subseteq \widehat{\mathcal{B}}^+$. Obviously Γ is a measurable countably separated section of the p. d. s. $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$. We calculate all irreducible *-representations associated with $\text{Orb}\chi$, $\chi \in \Gamma$.

- (i) Case $\chi_{0,\pm}$. The stabilizer of $\chi_{0,\pm}$ is trivial by Proposition 3.1, (ii). Put $\pi_\pm := \text{Ind}\chi_{0,\pm}$. We use Proposition 1.2 to compute the action of π_\pm on the orthonormal base $\{e_{-k}\}_{k \in \mathbb{N}_0}$.

$$\begin{aligned} \pi_\pm(b)e_{-k} &= (q^{2k}\chi_{0,\pm}(a) - q^{4k}\chi_{0,\pm}(a^2) + r)^{1/2} e_{-k+1} \\ &= (q^{2k}\lambda_\pm - (q^{2k}\lambda_\pm)^2 + r)^{1/2} e_{-k+1}, \\ \pi_\pm(b^*)e_{-k} &= \left(q^{2(k+1)}\lambda_\pm - (q^{2(k+1)}\lambda_\pm)^2 + r\right)^{1/2} e_{-k-1}, \\ \pi_\pm(a)e_{-k} &= \chi_{0,\pm}^{-k}(a) = q^{2k}\lambda_\pm e_{-k}. \end{aligned}$$

- (ii) Case χ_∞ . The stabilizer group H of χ is \mathbb{Z} . Let ρ be an irreducible *-representation of \mathcal{A}_H satisfying (1.6). Since $\chi(bb^* - b^*b) = 0$, we have $\rho(b)\rho(b^*) = \rho(b^*)\rho(b)$. By Schur's Lemma ρ is one-dimensional. For $\lambda \in \mathbb{C}$ such that $\lambda = \rho(b)$ we get $|\lambda|^2 = \rho(bb^*) = r$. Hence $\rho = \rho_\varphi$, for some $\varphi \in [0, 2\pi)$, where $\rho_\varphi(b) = e^{i\varphi}r^{1/2}$. Since $\mathcal{A}_H = \mathcal{A}$, $\pi_\varphi := \text{Ind}_{\mathcal{A}_H \uparrow \mathcal{A}}\rho_\varphi$ is equivalent to ρ_φ and we get

$$\pi_\varphi(a) = 0, \quad \pi_\varphi(b) = e^{i\varphi}r^{1/2}, \quad \pi_\varphi(b^*) = e^{-i\varphi}r^{1/2}, \quad \varphi \in [0, 2\pi).$$

By Theorem 1.4 these are all, up to unitary equivalence, irreducible *-representations of \mathcal{A} . Setting $\mathbf{e}_k := e_{-k}$, $k \in \mathbb{N}_0$, we see that these coincide with the ones found in [Pd]. In particular, we have the following

Theorem 3.2. *Every irreducible *-representation of the Podleś sphere $\mathcal{O}(S_{qr}^2)$, $q \in (0, 1)$, $r \in (0, \infty)$ is induced from a one-dimensional *-representation.*

In the remaining part of this section we describe the enveloping C*-algebra of $\mathcal{O}(S_{qr}^2)$. For let $(C_0(\widehat{\mathcal{B}}^+), \mathbb{Z}, \beta)$ be the C*-p.d.s. dual to $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$ as defined in the Subsection 0.2. Note, that the sets \mathcal{D}_k , $k \in \mathbb{Z}$, are compact, hence $I_k := C(\mathcal{D}_k)$. By definition of β we have

$$(3.4) \quad (\beta_k(f))(t) = f(\alpha_{-k}(t)) = f(q^{2k}t), \quad f \in I_{-k}, k \in \mathbb{Z}.$$

It is easily seen from the description of α in the Proposition 3.1 that the partial action $\beta = (\{I_k\}_{k \in \mathbb{Z}}, \{\beta_k\}_{k \in \mathbb{Z}})$ is defined by the partial automorphism $\Theta = (\theta, I, J)$, where $\theta = \beta_1$, $I = I_{-1}$, $J = I_1 = \mathcal{A}$.

Theorem 3.3. *The enveloping C*-algebra \mathfrak{A} of \mathcal{A} is isomorphic to the covariance algebra $C^*(C(\widehat{\mathcal{B}}^+), \Theta) \simeq C(\widehat{\mathcal{B}}^+) \times_\beta \mathbb{Z}$*

Proof. The proof goes similarly to the proof of the Theorem 2.6 by replacing the η -relation with ϵ -relation.

We first define *-homomorphism $\epsilon : \mathcal{A} \rightarrow C(\widehat{\mathcal{B}}^+) \times_\beta \mathbb{Z}$ by setting

$$\epsilon(a) = t \otimes 0, \quad \epsilon(b) = (q^2t - q^4t^2 + r)^{1/2} \otimes 1, \quad \epsilon(b^*) = (t - t^2 + r)^{1/2} \otimes (-1).$$

Direct computations using (3.4) show that $\epsilon(a), \epsilon(b), \epsilon(b^*)$ satisfy the defining relations of $\mathcal{O}(S_{qr}^2)$, that is, ϵ is well-defined. Every representation π of \mathcal{A} is bounded, hence well-behaved by Proposition 1.3. That is, π gives rise to a covariant representation $\pi|_{\mathcal{B}} \times u$, where u is the partial isometry in the polar decomposition $\pi(b) = uc$. On the other hand, every $*$ -representation of \mathfrak{A} is given by a covariant representation of the partial automorphism Θ . This proves the correspondence (0.3) for the representations of \mathcal{A} and \mathfrak{A} . \square

4. THE QUANTUM ALGEBRA $\mathcal{U}_q(su(2))$

In this section \mathcal{A} is the q -deformed enveloping $*$ -algebra $\mathcal{U}_q(su(2))$, $q > 0$, $q \neq 1$, which is generated by E, F, K, K^{-1} satisfying the following defining relations

$$KK^{-1} = K^{-1}K = \mathbf{1}, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,$$

$$[E, F] = EF - FE = \frac{K - K^{-1}}{q - q^{-1}},$$

$$E^* = FK, \quad F^* = K^{-1}E, \quad K^* = K.$$

In this section, we use the standard notation $[n] \equiv [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, where $n \in \mathbb{Z}$ and $q \neq 0$. Further X^0 denotes $\mathbf{1}$ if X is one of the four generators E, F, K, K^{-1} .

In [VS] the authors considered the family of irreducible $*$ -representations $\{\pi_{\omega, l} \mid \omega = \pm 1, l \in \frac{1}{2}\mathbb{N}_0\}$ of $\mathcal{U}_q(su(2))$. The representation $\pi_{\omega, l}$ acts on an orthonormal base $\{\mathbf{e}_m\}_{m=-l, \dots, l}$ of the representation space as follows:

$$(4.1) \quad \begin{aligned} \pi_{\omega, l}(K)\mathbf{e}_m &= \omega q^{2m}\mathbf{e}_m, \\ \pi_{\omega, l}(E)\mathbf{e}_m &= q^{m+1}\sqrt{[l-m][l+m+1]}\mathbf{e}_{m+1}, \\ \pi_{\omega, l}(F)\mathbf{e}_m &= \omega q^{-m}\sqrt{[l+m][l-m+1]}\mathbf{e}_{m-1}. \end{aligned}$$

We will show that every irreducible well-behaved representation of \mathcal{A} is unitarily equivalent to $\pi_{\omega, l}$ for some $l \in \frac{1}{2}\mathbb{N}_0$, $\omega = \pm 1$.

Define a \mathbb{Z} -grading of \mathcal{A} by setting $E \in \mathcal{A}_1$, $F \in \mathcal{A}_{-1}$ and $K, K^{-1} \in \mathcal{A}_0$. Then

$$\mathcal{B} := \mathcal{A}_0 = \text{Lin}\{F^l K^m E^l \mid l \in \mathbb{N}_0, m \in \mathbb{Z}\} = \text{Lin}\{E^l K^m F^l \mid l \in \mathbb{N}_0, m \in \mathbb{Z}\}.$$

The $*$ -subalgebra $\mathcal{B} \subseteq \mathcal{A}$ is commutative and is equal to $\mathbb{C}[EF, K, K^{-1}] = \mathbb{C}[C_q, K, K^{-1}]$. For $n \in \mathbb{N}_0$, we have

$$\begin{aligned} \mathcal{A}_n &= E^n \mathcal{B} = \text{Lin}\{E^{n+l} K^m F^l \mid l \in \mathbb{N}_0, m \in \mathbb{Z}\}, \\ \mathcal{A}_{-n} &= F^n \mathcal{B} = \text{Lin}\{F^{n+l} K^m E^l \mid l \in \mathbb{N}_0, m \in \mathbb{Z}\}. \end{aligned}$$

One can verify by a direct computation that the *quantum Casimir element* C_q is a central element in \mathcal{A} , where

$$C_q = EF + \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2}.$$

The following lemma can be easily proved by induction.

Lemma 4.1. *For every $n \in \mathbb{N}$ we have*

$$\begin{aligned} (i) \quad [E, F^n] &\equiv EF^n - F^n E = [n]F^{n-1}[K; 1-n], \\ (ii) \quad [E^n, F] &\equiv E^n F - FE^n = [n]E^{n-1}[K; n-1], \end{aligned}$$

where we set $[K; l] := (q^l K - q^{-l} K^{-1})/(q - q^{-1})$ for $l \in \mathbb{Z}$.

This lemma implies the following relations:

$$(4.2) \quad \begin{aligned} E^n F^n &= \prod_{j=1}^n (EF + [j-1][K; -j]), \\ F^n E^n &= \prod_{j=1}^n (EF - [j][K; j-1]), \quad n \in \mathbb{N}. \end{aligned}$$

Since $\mathcal{B} = \mathbb{C}[C_q, K, K^{-1}]$, every character $\chi \in \widehat{\mathcal{B}}$ is equal to some $\chi_{st} \in \widehat{\mathcal{B}}$, $(s, t) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}$ where

$$\chi_{st}(C_q) = s, \quad \chi_{st}(K) = t.$$

Proposition 4.2. (i) A character $\chi_{st} \in \widehat{\mathcal{B}}$ belongs to $\widehat{\mathcal{B}}^+$ if and only if

$$t = \pm q^{m-n} \text{ and } s = \frac{\pm q^{m+n+1} \pm q^{-m-n-1}}{(q - q^{-1})^2}, \text{ where } m, n \in \mathbb{N}_0.$$

In particular,

$$\widehat{\mathcal{B}}^+ = \{\chi_{m,n,+} \mid m, n \in \mathbb{N}_0\} \cup \{\chi_{m,n,-} \mid m, n \in \mathbb{N}_0\},$$

where $\chi_{m,n,\pm} = \chi_{st}$ with s, t from above.

(ii) The partial action $\alpha = (\{\mathcal{D}_n\}_{n \in \mathbb{Z}}, \{\alpha_n\}_{n \in \mathbb{Z}})$ is given as follows:

$$\mathcal{D}_{-k} = \{\chi_{m,n,\pm} \mid -m \leq k \leq n\}, \text{ and } \chi_{m,n,\pm}^k = \chi_{m+k,n-k,\pm}.$$

Proof. (i) Lemma 1.1 and Equations (4.2) imply that $\chi \in \widehat{\mathcal{B}}^+$ if and only if the following inequalities are satisfied for arbitrary $k \in \mathbb{N}$:

$$(4.3) \quad \chi(E^k F^k K^{-k}) = \prod_{j=1}^k \chi(EF + [j-1][K; -j]) \chi(K)^{-k} \geq 0.$$

$$(4.4) \quad \chi(F^k E^k K^k) = \prod_{j=1}^k \chi(EF - [j][K; j-1]) \chi(K)^k \geq 0,$$

We show that there exist $m, n \in \mathbb{N}_0$ such that

$$(4.5) \quad \chi(EF + [m][K; -m-1]) = 0,$$

$$(4.6) \quad \chi(EF - [n+1][K; n]) = 0.$$

Assume the contrary, i.e. $\chi(EF + [k][K; -k-1]) \neq 0$ for all $k \in \mathbb{N}_0$. Suppose $t > 0$. Then, by (4.3),

$$\chi(EF) \geq -[k] \chi([K; -k-1]) = \frac{(q^{-2k-1} - q^{-1})t + (q^{2k+1} - q)t^{-1}}{(q - q^{-1})^2}.$$

Such an value $\chi(EF) \in \mathbb{R}$ cannot exist, since $q^{-2k-1} \rightarrow \infty$ for $k \rightarrow \infty$ if $q \in (0, 1)$, respectively $q^{2k+1} \rightarrow \infty$ for $k \rightarrow \infty$ if $q > 1$. Analogously one obtains a contradiction for $t < 0$, using inequalities (4.3). Thus, $\chi(EF + [m][K; -m-1]) = 0$ for some $m \in \mathbb{N}_0$. Similarly one can prove that $\chi(EF - [n+1][K; n]) = 0$ for some

$n \in \mathbb{N}_0$, using inequalities (4.4). Subtracting (4.6) from (4.5) yields

$$\begin{aligned}
& [m]\chi([K; -m-1]) = -[n+1]\chi([K; n]) \\
\iff & (q^{-1} - q^{-2m-1})t - (q^{2m+1} - q)t^{-1} = \\
& = (q^{-1} - q^{2n+1})t - (q^{-2n-1} - q)t^{-1} \\
\iff & t^2 = \frac{q^{2m+1} - q^{-2n-1}}{q^{2n+1} - q^{-2m-1}} = \frac{q^{2m}(q - q^{-2m-2n-1})}{q^{2n}(q - q^{-2m-2n-1})} \\
\iff & t = \pm q^{m-n}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\chi(C_q) &= [n+1]\chi([K; n]) + \frac{q^{-1}\chi(K) + q\chi(K^{-1})}{(q - q^{-1})^2} \\
&= \frac{q^{2n+1}t + q^{-2n-1}t^{-1}}{(q - q^{-1})^2} = \frac{\pm q^{m+n+1} \pm q^{-m-n-1}}{(q - q^{-1})^2}.
\end{aligned}$$

(ii) Observe that $\chi_{m,n,\pm}(E^k F^k K^{-k}) \neq 0$ if and only if $k \leq m$ by (4.3) and (4.5). Analogously, $\chi_{m,n,\pm}(F^k E^k K^k) \neq 0$ if and only if $k \leq n$ by (4.4) and (4.6). This implies that $\chi_{m,n,\pm} \in \mathcal{D}_{-k}$ if and only if $-m \leq k \leq n$. Now suppose $k \in \{0, 1, \dots, n\}$. Since C_q commutes with E, F , we have

$$\begin{aligned}
\chi_{m,n,\pm}^k(K) &= \frac{\chi_{m,n,\pm}(E^{*k} K E^k)}{\chi_{m,n,\pm}(E^{*k} E^k)} = \frac{\chi_{m,n,\pm}(E^{*k} E^k q^{2k} K)}{\chi_{m,n,\pm}(E^{*k} E^k)} = q^{2k} \chi_{m,n,\pm}(K), \\
\chi_{m,n,\pm}^k(C_q) &= \frac{\chi_{m,n,\pm}(E^{*k} C_q E^k)}{\chi_{m,n,\pm}(E^{*k} E^k)} = \chi_{m,n,\pm}(C_q).
\end{aligned}$$

Analogously, if $k \in \{-m, -m+1, \dots, 0\}$ we have

$$\begin{aligned}
\chi_{m,n,\pm}^k(K) &= \frac{\chi_{m,n,\pm}(F^{*k} K F^k)}{\chi_{m,n,\pm}(F^{*k} F^k)} = q^{-2k} \chi_{m,n,\pm}(K), \\
\chi_{m,n,\pm}^k(C_q) &= \chi_{m,n,\pm}(C_q).
\end{aligned}$$

Hence, if $\chi_{m,n,\pm}^k$ is defined, then $\chi_{m,n,\pm}^k(K) = \pm q^{(m+k)-(n-k)} = \chi_{m+k,n-k,\pm}(K)$ and $\chi_{m,n,\pm}^k(C_q) = \chi_{m,n,\pm}(C_q)$. \square

In particular, the previous proposition shows that for each $\chi \in \widehat{\mathcal{B}}^+$ the stabilizer $\text{St}\chi$ is trivial. We set

$$\Gamma := \{\chi_{0,n,+} \mid n \in \mathbb{N}_0\} \cup \{\chi_{n,-} \mid n \in \mathbb{N}_0\}.$$

As in Section 2, we conclude that Γ is a measurable countably separated section of the partial action. Using Proposition 4.2 we conclude that $\text{Orb}\chi_{0,n,\pm}$ consists of $n+1$ elements and hence $\text{Ind}\chi_{0,n,\pm}$ has dimension $n+1$ by Proposition 1.2, where $\chi_{0,n,\pm} \in \Gamma$. Now put $l := \frac{n}{2}$ and $\pi_{\omega,l} := \text{Ind}\chi_{0,n,\pm}$, where $\omega = \pm 1$. Let $\{e_{l+m}\}_{m=-l, -l+1, \dots, l}$ be an orthonormal base of the representation space $\mathcal{H}_{\pi_{l,\pm}}$ of $\pi_{l,\pm}$. For notational convenience, we put $e_{l+1} := 0$, and $e_{-l-1} := 0$.

Using Proposition 1.2, relations (4.2), Proposition 4.2 and the facts that $\chi_{0,n,\pm}(EF) = 0$, $\chi_{0,n,\pm}([K; l+m]) = -\omega[l-m]$, we obtain the action of $\pi_{\omega,l}$

on the base vectors e_{l+m} .

$$\begin{aligned}
\pi_{\omega,l}(K)e_{l+m} &= \chi_{0,n,\pm}^{l+m}(K)e_{l+m} = \chi_{l+m,n-l-m,\pm}(K)e_{l+m} = \omega q^{2m}e_{l+m}, \\
\pi_{\omega,l}(E)e_{l+m} &= \frac{\chi_{0,n,\pm}(E^{*(l+m+1)}E^{l+m+1})}{(\chi_{0,n,\pm}(E^{*(l+m+1)}E^{l+m+1})\chi_{0,n,\pm}(E^{*(l+m)}E^{l+m}))^{1/2}}e_{l+m+1} \\
&= \left(\frac{\chi_{0,n,\pm}(E^{*(l+m+1)}E^{l+m+1})}{\chi_{0,n,\pm}(E^{*(l+m)}E^{l+m})} \right)^{1/2} e_{l+m+1} \\
&= \left(q^{(l+m+1)(l+m+2)-(l+m)(l+m+1)} \right)^{1/2} \\
&\quad \times (\chi_{0,n,\pm}(EF - [l+m+1][K; l+m])\chi_{0,n,\pm}(K))^{1/2} e_{l+m+1} \\
&= q^{m+1} \sqrt{[l-m][l+m+1]} e_{l+m+1}, \\
\pi_{\omega,l}(F)e_{l+m} &= \pi_{\omega,l}(E^*K^{-1})e_{l+m} = \omega q^{-2m} \pi_{\omega,l}(E)^* e_{l+m} \\
&= \omega q^{-m} \sqrt{[l+m][l-m+1]} e_{l+m-1}.
\end{aligned}$$

Putting $\mathbf{e}_m := e_{l+m}$, $m = -l, \dots, l$, we see that all irreducible well-behaved *-representations of the quantum algebra $\mathcal{U}_q(su(2))$ are unitarily equivalent to the irreducible well-behaved *-representation $\pi_{\omega,l}$, given by the formulas (4.1), for some $\omega \in \{-1, +1\}$ and $l \in \frac{1}{2}\mathbb{N}_0$. In particular, all irreducible *-representations of $\mathcal{U}_q(su(2))$ are bounded. Summarizing the above discussion, we obtain the following

Theorem 4.3. *Every irreducible well-behaved *-representation of $\mathcal{U}_q(su(2))$, $q \in \mathbb{R}^+ \setminus \{1\}$, is induced from a one-dimensional *-representation.*

Similarly to Lemma 2.3 and Theorem 2.5 one can prove the following Lemma and Theorem.

Lemma 4.4. *The polynomial*

$$(EF - [2][K; 1])(EF - [3][K; 2]) \in \mathbb{C}[EF, K, K^{-1}]$$

*is positive in every well-behaved *-representation of \mathcal{A} and is not of the form $\sum_{k=1}^n a_k^* a_k$ for $a_k \in \mathcal{A}$.*

Theorem 4.5. *There exists a *-representation of $\mathcal{U}_q(su(2))$, $q \in \mathbb{R}^+ \setminus \{1\}$, which has no well-behaved extension in a possibly larger Hilbert space.*

Let $\text{Rep}\mathcal{A}$ denote the category of well-behaved non-degenerate representations of \mathcal{A} . Then \mathcal{A} , considered with $\text{Rep}\mathcal{A}$ and generators E, F, K, K^{-1} , has a C^* -envelope \mathfrak{A} in the sense of Definition 0.1. As in Section 2, let $(C_0(\widehat{\mathcal{B}}^+), \mathbb{Z}, \beta)$ be the C^* -p.d.s. dual to $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$. The description of the p.d.s. $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$ in Proposition 4.2 implies that the C^* -p.d.s. $(C_0(\widehat{\mathcal{B}}^+), \mathbb{Z}, \beta)$ is defined by the partial automorphism $\Theta = (\theta, I, J)$, where $I = I_{-1}$, $J = I_1$, $(\theta(f))(t) = (\beta_1(f))(t)$, $f \in I_{-1}$.

The proof of the following theorem is completely analogous to the proof of Theorem 2.6.

Theorem 4.6. *Consider the q -deformed enveloping algebra $\mathcal{A} = \mathcal{U}_q(su(2))$ with generators E, F, K, K^{-1} and the category of well-behaved representations $\text{Rep}\mathcal{A}$. Then the covariance algebra $\mathfrak{A} := C^*(C_0(\widehat{\mathcal{B}}^+), \Theta)$ is a C^* -envelope of \mathcal{A} .*

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